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Advanced Linear Algebra (MA 409)
Problem Sheet - 3

Linear Combinations and Systems of Linear Equations

1. Label the following statements as true or false.
   (a) The zero vector is a linear combination of any nonempty set of vectors.
   (b) The span of $\varnothing$ is $\varnothing$.
   (c) If S is a subset of a vector space V, then span(S) equals the intersection of all subspaces of V that contain S.
   (d) In solving a system of linear equations, it is permissible to multiply an equation by any constant.
   (e) In solving a system of linear equations, it is permissible to add any multiple of one equation to another.
   (f) Every system of linear equations has a solution.

2. For each of the following lists of vectors in $\mathbb{R}^3$, determine whether the first vector can be expressed as a linear combination of the other two.
   (a) $(-2,0,3),(1,3,0),(2,4,-1)$
   (b) $(3,4,1),(1,-2,1),(-2,-1,1)$
   (c) $(-2,2,2),(1,2,-1),(-3,-3,3)$

3. For each list of polynomials in $P_3(\mathbb{R})$, determine whether the first polynomial can be expressed as a linear combination of the other two.
   (a) $4x^3+2x^2-6, x^3-2x^2+4x+1, 3x^3-6x^2+x+4$
   (b) $x^3+x^2+2x+13, 2x^3-3x^2+4x+1, x^3-x^2+2x+3$
   (c) $6x^3-3x^2+x+2, x^3-x^2+2x+3, 2x^3-3x+1$

4. In each part, determine whether the given vector is in the span of S.
   (a) $(-1,2,1), S=\{(1,0,2),(-1,1,1)\}$
   (b) $(2,-1,1,-3), S=\{(1,0,1,-1),(0,1,1,1)\}$
   (c) $2x^3-x^2+x+3, S=\{x^3+x^2+x+1, x^2+x+1, x+1\}$
   (d)
   $$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S=\left\{\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\right\}$$

5. Show that the vectors $(1,1,0),(1,0,1)$, and $(0,1,1)$ generate F^3.

6. In F^n, let e_j denote the vector whose jth coordinate is 1 and whose other coordinates are 0. Prove that $\{e_1, e_2, \ldots, e_n\}$ generates F^n.

7. Show that $P_n(F)$ is generated by $\{1, x, x^2, \ldots, x^n\}$.

8. Show that the matrices
   $$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
   generate $M_{2\times 2}(F)$.

9. Show that if
$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
then the span of $\{M_1, M_2, M_3\}$ is the set of all symmetric 2×2 matrices.

10. Show that a subset W of a vector space V is a subspace of V if and only if span(W) = W.

11. Show that if S_1 and S_2 are subsets of a vector space V such that $S_1 \subseteq S_2$, then $\text{span}(S_1) \subseteq \text{span}(S_2)$. In particular, if $S_1 \subseteq S_2$ and span(S_1) = V, deduce that span(S_2) = V.

12. Show that if S_1 and S_2 are arbitrary subsets of a vector space V, then $\text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2)$.

13. Let S_1 and S_2 be subsets of a vector space V. Prove that $\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \cap \text{span}(S_2)$. Give an example in which $\text{span}(S_1 \cap S_2)$ and $\text{span}(S_1) \cap \text{span}(S_2)$ are equal and one in which they are unequal.

14. Let V be a vector space and S a subset of V with the property that whenever $v_1, v_2, \ldots, v_n \in S$ and $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$, then $a_1 = a_2 = \cdots = a_n = 0$. Prove that every vector in the span of S can be uniquely written as a linear combination of vectors of S.

15. Let W be a subspace of a vector space V. Under what conditions are there only a finite number of distinct subsets S of W such that S generates W?

16. (a) Consider the vector $x_1 = (1, 3, 2)$ and $x_2 = (-2, 4, 3)$ in $\mathbb{R}^3$. Show that the span of $\{x_1, x_2\}$ is $\Big\{(\xi_1, \xi_2, \xi_3) : \xi_1 - 7\xi_2 + 10\xi_3 = 0\Big\} = \Big\{(\alpha, \beta, \frac{-\alpha+7\beta}{10}) : \alpha, \beta \in \mathbb{R}\Big\}$.

    (b) Consider the vectors $x_1 = (1, 2, 1, -1), x_2 = (2, 4, 1, 1), x_3 = (-1, -2, -2, -4)$ and $x_4 = (3, 6, 2, 0)$ in $\mathbb{R}^4$. Show that the span of $\{x_1, x_2, x_3, x_4\}$ is $\Big\{(\xi_1, \xi_2, \xi_3, \xi_4) : 2\xi_1 - \xi_2 = 0, 2\xi_1 - 3\xi_3 - \xi_4 = 0\Big\}$. Show that this subspace can also be written as $\Big\{(\alpha, 2\alpha, \beta, 2\alpha - 3\beta) : \alpha, \beta \in \mathbb{R}\Big\}$.

17. Consider the vector space $\mathbb{R}$ over $\mathbb{Q}$.

    (a) For the subspace $\mathbb{Q}$, show that any subset containing a non-zero vector is a generating set.

    (b) For the subspace $\{\alpha + \beta\sqrt{2} + \gamma\sqrt{3} : \alpha, \beta, \gamma \in \mathbb{Q}\}$, show that $\{1, \sqrt{2}, \sqrt{3}\}$ is a generating set.

    (c) For the vector space $\mathcal{F}(X, F)$ show that the set $\{f : 0 \in \text{range of } f\}$ is a generating set provided X has at least two elements and $\{f : 0 \notin \text{range of } f\}$ is a generating set provided F has at least 3 elements.

    (d) In the vector space $P(\Omega)$ over $\mathbb{Z}_2$ with $\Omega = \{1, 2, \ldots, 5\}$, find the subspace generated by $\Big\{\{1, 2, 3\}, \{2, 3, 4\}$

18. (a) For any two subsets A and B of a vector space V, show that

    (i) $Sp(A) \cup Sp(B) \subseteq Sp(A \cup B)$,

    (ii) $Sp(A \cap B) \subseteq Sp(A) \cap Sp(B)$, and that proper inclusion is possible in each.

    (b) Prove or disprove : $Sp(A) \cap Sp(B) \neq \{0\} \implies A \cap B \neq \varnothing$.

19. Let W be a subspace of a vector space V. If x and y are vectors such that $x + y \in W$ then show that either both x and y belong to W or none of x and y belongs to W. If x is a vector and α is a non-zero scalar such that $\alpha x \in W$ then show that $x \in W$. What can you say if $x + y + z \in W$?

20. Show that the intersection of any family of subspaces is a subspace (the intersection of the empty family of subsets of V is defined to be V).

21. Show that the for any set $A \subseteq V$, Sp(A) is the intersection of all subspaces of V containing A.

22. Let W_1 and W_2 be subspaces of V. Then prove that $W_1 \cup W_2$ is a subspace iff either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. (Hint for only if part : If $W_1 \nsubseteq W_2$ and $W_2 \nsubseteq W_1$, consider $x + y$ where $x \in W_1 - W_2$ and $y \in W_2 - W_1$).

23. Write down the 6 different subspaces of F^2 where F is the field of residues mod 3, denoted by $\mathbb{Z}_3$. Draw a figure but note that geometric intuition may not be entirely correct here.

24. Let X be the set of all positive integers. In the vector space $\mathcal{F}(X, \mathbb{R})$, what is the span of the set $A = \{f_i : i \geq 1\}$, where f_i is the function in $\mathcal{F}(X, \mathbb{R})$ taking value 1 at $x = i$ and 0 elsewhere? Show that if $f \in Sp(A)$ then the range of f is finite but the converse is not true.

System of Linear Equations

Three types of operations to simply the original system:

(a) interchanging the order of any two equations in the system ;

(b) multiplying any equation in the system by a nonzero constant ;

(c) adding a constant multiple of any equation to another equation in the system.

Note that these operations do not change the set of solutions to the original system. We employ these (a), (b) and (c) to obtain a system of equations that has the following properties :

(1) The first non-zero coefficient in each equation is one.

(2) If an unknown is the first unknown with a non-zero coefficient in some equation, then that unknown occurs with a zero coefficient in each of the other equations.

(3) The first unknown with a non-zero coefficient in any equation has a larger subscript than the first unknown with a non-zero coefficient in any preceding equation.

Once a system with properties (1), (2), and (3) has been obtained, it is easy to solve for some of the unknowns in terms of the others. If, however, in the course of using operations (1), (2), and (3) a system containing an equation of the form $0 = c$, where c is nonzero, is obtained, then the original system has no solutions.

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25. Solve the following systems of linear equations by reducing to a system of equations that have the properties (1), (2), (3).

(a)

$$\begin{aligned} 2x_1 - 2x_2 - 3x_3 &= -2 \\ 3x_1 - 3x_2 - 2x_3 + 5x_4 &= 7 \\ x_1 - x_2 - 2x_3 - x_4 &= -3 \end{aligned}$$

(b)

$$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 &= 5 \\ x_1 + 4x_2 - 3x_3 - 3x_4 &= 6 \\ 2x_1 + 3x_2 - x_3 + 4x_4 &= 8 \end{aligned}$$

(c)

$$\begin{aligned} x_1 + 2x_2 + 6x_3 &= -1 \\ 2x_1 + x_2 + x_3 &= 8 \\ 3x_1 + x_2 - x_3 &= 15 \\ x_1 + 3x_2 + 10x_3 &= -5 \end{aligned}$$

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